Published by Institute of Physics Publishing for SISSA/ISAS
Received: *March 12, 2004*
Accepted: *April 22, 2004*

Nonabelian gauge field and dual description of fuzzy sphere

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Abstract: In matrix models, higher dimensional D-branes are obtained by imposing a noncommutative relation to coordinates of lower dimensional D-branes. On the other hand, a dual description of this noncommutative space is provided by higher dimensional D-branes with gauge fields. Fuzzy spheres can appear as a configuration of lower dimensional D-branes in a constant R-R field strength background. In this paper, we consider a dual description of higher dimensional fuzzy spheres by introducing nonabelian gauge fields on higher dimensional spherical D-branes. By using the Born-Infeld action, we show that a fuzzy $2k$-sphere and spherical D$2k$-branes with a nonabelian gauge field whose Chern character is nontrivial are the same objects when n is large. We discuss a relationship between the noncommutative geometry and nonabelian gauge fields. Nonabelian gauge fields are represented by noncommutative matrices including the coordinate dependence. A similarity to the quantum Hall system is also studied.

Keywords: Solitons Monopoles and Instantons, D-branes, M(atrix) Theories, Non-Commutative Geometry.

 http://jhep.sissa.it/archive/papers/jhep042004058/jhep042004058.pdf

Contents

1. Introduction

String theory is considered as the best candidate for a quantum theory of gravity or a unified theory of all interactions. Much attention has been given to noncommutative geometry because it is expected to capture some aspects of the quantum gravity. Over the past few years several papers have been devoted to the study of a relationship between noncommutative geometry and string theory. The need of noncommutative geometry in string theory is easily understood by considering a world-volume action of D-branes. D-branes are defined as the endpoints of open strings. Since gauge fields appear in the ground state of open strings, the low energy dynamics of D-branes is described by gauge fields. One of the most interesting aspects is the appearance of nonabelian gauge symmetry from the world-volume theory of some coincident D-branes. A low energy effective action of N D-branes is provided by the dimensional reduction of the ten-dimensional $\mathrm{U}(N)$ Yang-Mills theory, and transverse coordinates of N D-branes are expressed by $\mathrm{U}(N)$ adjoint scalars [1]. The appearance of this matrix-valued coordinate indicates relationships between string theory and noncommutative geometry.

Concrete models for studying the nonperturbative properties of string theory are proposed in [2, 3]. These matrix models are constructed by taking lower dimensional D-branes as fundamental degrees of freedom. Since higher dimensional D-branes also exist in string

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theories, they need to be contained in matrix models. It is important to understand relationships between D-branes of different dimensions because higher dimensional ones are constructed form lower dimensional ones. A study of a world-volume action for D-branes can reveal the relationship. The Chern-Simons coupling indicates an interesting relationships between D-branes [4]–[7]. Let us consider a matrix model of D0-branes as a simple example. In this model, a D2-brane is represented by two noncommutative matrices,

$$[X_1, X_2] = -iC_{12}1\,, \tag{1.1}$$

where X_i are transverse coordinates of D0-branes [8, 2]. The commutator measures the charge of D2-brane. In general, we can say that higher dimensional D-branes are expressed by noncommutative geometry from lower dimensional D-branes. What has to be noticed is that the above matrix configuration is not just a D2-brane but a bound state of D0-branes and a D2-brane. In the world-volume theory of a D2-brane, D0-branes bounded on it are expressed by an abelian gauge field configuration with a nonzero first Chern class, which is supported by the following coupling:

$$\mu_2 \int_{2+1} \left(C^{(3)} + C^{(1)} \wedge F \right), \tag{1.2}$$

where $C^{(k)}$ represents R-R k-form field. This coupling implies that a gauge field on a D2-brane couples to the R-R one-form field, which is associated with D0-branes. Therefore the first Chern class provides the charge of D0-branes. This fact can be generalized to the fact that lower dimensional D-branes bounded on higher dimensional D-branes are represented by gauge fields which is topologically nontrivial. We then find that there are two descriptions for a system. If we start with lower dimensional D-branes, higher dimensional D-branes are obtained by imposing a noncommutative relation to coordinates of lower dimensional D-branes. On the other hand, on the world-volume theory of higher dimensional D-branes, a gauge field with a nonvanishing Chern number is needed to express lower dimensional D-branes. See also [9] for this correspondence. These facts are extensively reviewed in [10, 11, 12].

In the previous paragraph, we showed an example of two dual descriptions for a bound state of a flat D2-brane and D0-branes. A noncommutative plane is dual to a flat D2-brane with an abelian gauge field. It is important to study two descriptions because the second description gives another viewpoint for noncommutative geometry. This study helps us to understand the role of noncommutative geometry in string theory. This correspondence is also expected to hold for curved D-branes. A simple generalization is to consider the fuzzy sphere. A fuzzy two-sphere has a dual description in terms of an abelian gauge field on a spherical D2-brane, and is interpreted as a bound state of a spherical D2-brane and D0-branes [13, 7, 14, 15, 16, 17]. We need to emphasize that gauge fields are abelian for both cases. A higher dimensional generalization is an interesting subject since a higher dimensional fuzzy sphere has some characteristic features. It is first constructed in [18], and further analyses revealed the interesting structures of higher dimensional fuzzy sphere [19, 20, 21]. See also [22]–[26]. It is pointed out in [20] that a fuzzy $2k$-sphere is expressed by a coset space $\mathrm{SO}(2k+1)/\,\mathrm{U}(k)$, and therefore the dimension is not $2k$ but $k(k+1)$. The number of the extra dimensions makes the higher dimensional fuzzy sphere complicated. As analyzed in [27] for the $k = 2$ case, a nonabelian gauge field is needed to realize a dual description of higher dimensional fuzzy spheres. The use of a nonabelian gauge field is closely related to the existence of the extra dimensions. A dual aspect of higher dimensional fuzzy spheres is partially discussed in [27, 20, 53, 46]. In this paper, we explicitly construct a nonabelian gauge field to realize a dual description of higher dimensional fuzzy spheres and compare two descriptions by using the Born-Infeld action. We see that two descriptions provide the same result when the size n of matrices realizing the fuzzy sphere is enough large. Such a analysis shows an interesting relationship between the noncommutativity on the fuzzy sphere and the nonabelian gauge field. See also a recent work [28] which calculates D-brane charges for various noncommutative configurations.

The organization of this paper is as follow. In section 2, we review the algebra of the fuzzy $2k$-sphere. Some distinctive aspects of the higher dimensional fuzzy sphere are explained. We consider N D0-branes in a constant R-R $(2k+2)$-form field strength background in section 3. An action of D0-branes in the background is described by a matrix model with the Chern-Simons term. We see that N D0-branes form a fuzzy $2k$-sphere at a classical extremum of the matrix model. This phenomenon is called the dielectric effect [7]. We calculate the value of the potential for the fuzzy sphere. Since higher dimensional fuzzy spheres are realized at a local maximum of the classical potential, they are classically unstable in this situation. To understand a dual description of higher dimensional fuzzy spheres is a main part of this paper. We consider a world-volume theory of spherical D$2k$-branes in section 4. A dual description of a fuzzy $2k$-sphere is expected to be given by a bound state of spherical D$2k$-branes and D0-branes. The D0-branes are represented by a gauge field configuration with a nonzero Chern number. Such gauge fields are recently used to construct a higher dimensional quantum Hall system in [29]–[32]. An interesting fact for the higher dimensional case is that the gauge field is nonabelian and expressed by using matrices which are associated with a lower dimensional fuzzy sphere. This was also suggested in [21] by studying a relationship between a higher dimensional fuzzy sphere and a lower dimensional fuzzy sphere. A world-volume action of spherical D$2k$-branes with a (nonabelian) gauge field shows two extrema, one of them corresponds to a fuzzy $2k$-sphere in the matrix model description. We compare some quantities such as the potential, the radius of sphere and the charge of lower dimensional D-branes in two descriptions, and it is shown that these two descriptions coincide at large-n. Accordingly we conclude that a fuzzy $2k$-sphere and spherical D$2k$-branes with an SO$(2k)$ nonabelian gauge field are the same objects at large-n. In section 5, we discuss a relationship between the noncommutativity and nonabelian gauge fields. It is explained that a nonabelian gauge field is described by a matrix including the coordinate dependence. We can observe an interesting mixing between the noncommutativity of coordinates and that of nonabelian gauge fields. We also provide an explanation about the large-n limit by studying the lowest Landau level condition in a higher dimensional quantum Hall system. A relation to the zero slope limit in [47] is also discussed. Section 6 devotes to summary and discussions. In appendix A, we summarize some formulae of the fuzzy sphere. In appendix B, we review the construction of monopole gauge fields on even dimensional spheres.

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2. Fuzzy sphere

In this section, we review the algebra of fuzzy sphere in diverse dimensions. Our interest in this paper is restricted to even dimensional spheres. Odd-dimensional fuzzy spheres are investigated in [33, 34].

We first explain the fuzzy two-sphere [35, 36, 13, 37, 38]. A coordinate of a fuzzy two-sphere is given by the SU(2) algebra;

$$[X_\mu, X_\nu] = 2i\alpha\epsilon_{\mu\nu\lambda}X_\lambda\,, \qquad X_\mu = \alpha L_\mu\,, \tag{2.1}$$

where L_μ denotes the spin $n/2$-representation of SU(2), and α is a dimensionful constant. n can take any positive integers. The quadratic Casimir of SU(2) provides the radius of the two-sphere;

$$r^2 \equiv X_\mu X_\mu = \alpha^2 L_\mu L_\mu = \alpha^2 n(n+2)1_{n+1}\,. \tag{2.2}$$

In this realization, the size $(n+1)$ of the matrix is interpreted as the number of D0-branes.

Let us now consider the place which is labelled by $L_3 = n$, which corresponds to the north pole of a two-sphere. Around this point, the fuzzy two-sphere algebra (2.1) becomes a noncommutative plane after we take a large-n limit,

$$[X_\mu, X_\nu] = 2i\alpha^2 n\epsilon_{\mu\nu}1\,. \tag{2.3}$$

In this sense, the fuzzy sphere algebra is considered as a generalization of a noncommutative plane.

We next review the algebra of higher dimensional fuzzy spheres [18]–[21]. It is natural to start with the SO$(2k+1)$ algebra since it is a symmetry of a $2k$-sphere,

$$[\hat{G}_{\mu\nu}, \hat{G}_{\lambda\rho}] = 2\left(\delta_{\nu\lambda}\hat{G}_{\mu\rho} + \delta_{\mu\rho}\hat{G}_{\nu\lambda} - \delta_{\mu\lambda}\hat{G}_{\nu\rho} - \delta_{\nu\rho}\hat{G}_{\mu\lambda}\right). \tag{2.4}$$

An important fact is that any representations of this algebra do not always construct a fuzzy sphere. We have to consider a representation whose highest weight state is labelled by $[0,\ldots,0,n]$, where n is a positive integer. We have used the Dynkin index to label the representation. This representation is considered as a generalization of the spinor representation since $\hat{G}_{\mu\nu}$ reduces to $\Gamma_{\mu\nu}$ when $n=1$. Note that $\hat{G}_{\mu\nu}$ is explicitly constructed from $\Gamma_{\mu\nu}$ by using a symmetric tensor product as in [18, 19] (see also [50] for the detailed calculation).

We denote the size of the matrix representation by N_k. It is calculated as [19]:

$$\begin{aligned} N_1 &= n+1\,, \qquad N_2 = \frac{1}{6}(n+1)(n+2)(n+3)\,, \\ N_3 &= \frac{1}{360}(n+1)(n+2)(n+3)^2(n+4)(n+5)\,, \\ N_4 &= \frac{1}{302400}(n+1)(n+2)(n+3)^2(n+4)^2(n+5)^2(n+6)(n+7)\,. \end{aligned} \tag{2.5}$$

The $k=1$ case is included in this table. A big difference between a fuzzy two-sphere and a fuzzy $2k$-sphere $(k \neq 1)$ is that N_1 can take any positive integers while N_k $(k \neq 1)$ cannot. We provide some comments on this point in section 6.

Coordinates of a $2k$-sphere are included in $\hat{G}_{\mu\nu}$.[1] It is convenient to introduce $\hat{G}_\mu$ which satisfy

$$[\hat{G}_\mu, \hat{G}_\nu] = 2\hat{G}_{\mu\nu}\,,$$
$$[\hat{G}_\mu, \hat{G}_{\nu\lambda}] = 2\left(\delta_{\mu\nu}\hat{G}_\lambda - \delta_{\mu\lambda}\hat{G}_\nu\right). \tag{2.6}$$

When $n=1$, $\hat{G}_\mu$ becomes the $(2k+1)$-dimensional gamma matrix Γ_μ. We can define coordinates of a fuzzy $2k$-sphere as $X_\mu = \alpha\hat{G}_\mu$ because of the following relation

$$\hat{G}_\mu\hat{G}_\mu = n(n+2k)1_{N_k}\,. \tag{2.7}$$

The radius of a fuzzy $2k$-sphere is

$$r^2 = \alpha^2 n(n+2k)\,. \tag{2.8}$$

Other relations for $\hat{G}_\mu$ and $\hat{G}_{\mu\nu}$ are summarized in appendix A.

$\hat{G}_\mu$ and $\hat{G}_{\mu\nu}$ really form the SO$(2k+2)$ algebra. If we define $\Sigma_{\mu\nu} \equiv \hat{G}_{\mu\nu}$ and $\Sigma_{2k+2,\mu} \equiv i\hat{G}_\mu$, Σ_{ab} $(a,b=1,\ldots,2k+2)$ satisfy the SO$(2k+2)$ algebra, belonging to an irreducible spinor representation of SO$(2k+2)$.

Let us define a noncommutative scale on a fuzzy sphere. A commutator of coordinates is given by $\alpha^2\hat{G}_{\mu\nu}$, and the order of $\hat{G}_{\mu\nu}$ is n due to the relation (A.3). Therefore a noncommutative scale l_{nc} can be defined as follows,

$$l_{nc}^2 \simeq \alpha^2 n \simeq \frac{r^2}{n}\,. \tag{2.9}$$

We now comment on the number of independent matrices. $\hat{G}_\mu$ and $\hat{G}_{\mu\nu}$ have $(2k+1)$ and $k(2k+1)$ components respectively. Although there are totally $(k+1)(2k+1)$ components, each component is not independent because of some constraints between them. The number of independent components is really given by $k(k+1)$. This is understood by considering the fact that the fuzzy $2k$-sphere is given by the coset space SO$(2k+1)/$U(k) [20]. The difference between the fuzzy sphere which is given by SO$(2k+1)/$U(k) and the usual sphere by SO$(2k+1)/$SO$(2k)$ is important. It is shown in [20] that a fuzzy sphere has a bundle structure over an usual sphere. Accordingly higher dimensional fuzzy spheres have some extra dimensions. Coordinates of the extra dimensions are also noncommutative, and a noncommutative scale of the extra space is also given by (2.9). In the remainder of this paper, we clarify this unusual aspect by considering the dual description of the fuzzy sphere.

3. D0-branes under R-R field strength background

In this section, we consider a collection of D0-branes in a constant R-R field strength background. A coupling of D0-branes with the R-R field strength background induces an interesting effect. D0-branes are polarized into a higher dimensional noncommutative

[1] $i\hat{G}_{5a}$ are naturally identified with coordinates around the north pole.

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geometry, which is called the dielectric effect [7]. A low energy dynamics of D0-branes in such a background is described by a matrix action. We will see that transverse coordinates of D0-branes form a fuzzy sphere at an extremum of the model. Some works investigating higher dimensional D-branes in such an background have been reported in [39, 40].

The tension and the charge of a Dp-brane are defined as

$$T_p = \mu_p = \frac{2\pi}{g(2\pi l_s)^{p+1}} . \tag{3.1}$$

We also define $\lambda \equiv 2\pi l_s^2$. In this notation, the charge of a Dp-brane is related to that of a D0-brane as $\mu_0 = (2\pi\lambda)^{p/2}\mu_p$. Basically we follow the notation of [7].

The dynamics of a world-volume theory of D-branes is described by the Born-Infeld action [41] and the Chern-Simons action [4, 5]. The case of nonabelian was developped in [6, 7]. The Born-Infeld action for N D0-branes in a flat space, with all other fields except transverse scalar fields vanishing, is given by

$$S_{BI} = -T_0 \int dt \,\mathrm{Tr} \sqrt{det(Q^i_j)} \simeq -T_0 \int dt \left(N - \frac{\lambda^2}{4} \mathrm{Tr}[\Phi_i, \Phi_j][\Phi_i, \Phi_j] \right) , \tag{3.2}$$

where we have expanded the square root by assuming the condition $\lambda[\Phi_i, \Phi_j] \ll 1$. This action can describe a low energy dynamics of N D0-branes in a flat space. Φ_i is an $N \times N$ matrix-valued coordinate whose dimension is $length^{-1}$, representing a transverse motion of N D0-branes. We define a coordinate whose dimension is $length$ as $X_i = \lambda\Phi_i$. The second term in (3.2) is also obtained form the dimensional reduction of the ten-dimensional U(N) super Yang-Mills action.

We next consider the Chern-Simons term. A coupling of N D0-branes to the R-R potential is given [6, 7] by

$$\begin{aligned} S_{CS} &= \mu_0 \int \mathrm{Tr} \left(P[e^{i\lambda i_\Phi i_\Phi} \sum C^{(n)}] \right) \qquad (3.3) \\ &= \mu_0 \int \mathrm{Tr} \left(P\left[C^{(1)} + i\lambda i_\Phi i_\Phi C^{(3)} - \frac{\lambda^2}{2}(i_\Phi i_\Phi)^2 C^{(5)} - i\frac{\lambda^3}{3!}(i_\Phi i_\Phi)^3 C^{(7)} + \frac{\lambda^4}{4!}(i_\Phi i_\Phi)^4 C^{(9)} \right] \right) . \end{aligned}$$

In this action, we considered a case where the NS-NS two-form field B and the gauge field strength F vanish. The notation $P[\cdots]$ denotes the pull back of the spacetime tensor to the world-volume, and acts on $C^{(1)}$ as

$$P[C^{(1)}]_t = C_\mu \frac{\partial X_\mu}{\partial t} = C_t + \lambda C_i \frac{\partial \Phi_i}{\partial t} . \tag{3.4}$$

We must draw attention to the fact that the R-R fields are functional of transverse scalar fields Φ. The dependence is actually defined by using a non-abelian Taylor expansion [42];

$$C(t, \Phi) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \Phi^{i_1} \cdots \Phi^{i_n} (\partial_{x_{i_1}} \cdots \partial_{x_{i_n}}) C(t, x) \mid_{x=0} . \tag{3.5}$$

The symbol i_Φ is introduced in [7], and denotes the following operation,

$$i_\Phi i_\Phi C^{(2)} = \Phi^j \Phi^i C^{(2)}_{ij} . \tag{3.6}$$

We now consider a case where only a constant R-R $(2k+2)$-form field strength is nonzero:

$$F^{(2k+2)}_{ti_1\ldots i_{2k+1}} = f_k \epsilon_{i_1\ldots i_{2k+1}} , \tag{3.7}$$

where f_k is a constant and determined later. Then we find that the leading order interaction term is obtained from the Chern-Simons term (3.3) as

$$\begin{aligned} S_{CS} &= -\frac{1}{(2k+1)}\frac{i^k}{k!}\lambda^{k+1}\mu_0 \int dt\, \mathrm{Tr}(\Phi_{i_1}\cdots\Phi_{i_{2k+1}})F^{(2k+2)}_{ti_1\ldots i_{2k+1}} \\ &= -\frac{1}{(2k+1)}\frac{i^k}{k!}\lambda^{k+1} f_k \mu_0 \int dt\, \mathrm{Tr}(\Phi_{i_1}\cdots\Phi_{i_{2k+1}})\epsilon_{i_1\ldots i_{2k+1}} . \end{aligned} \tag{3.8}$$

By combining (3.2) with (3.8), a low energy effective scalar potential of N D0-branes in a constant R-R $(2k+2)$-form field strength background is found to be

$$V = \lambda^2 T_0 \left(-\frac{1}{4}\,\mathrm{Tr}[\Phi_i,\Phi_j][\Phi_i,\Phi_j] + \frac{1}{(2k+1)}\frac{i^k}{k!}\lambda^{k-1} f_k\, \mathrm{Tr}(\Phi_{i_1}\cdots\Phi_{i_{2k+1}})\epsilon_{i_1\ldots i_{2k+1}} \right) , \tag{3.9}$$

where we have ignored the energy of N D0-branes, NT_0, which is not important in this discussion. The indices i, j run over $1, 2, \ldots, 2k+1$. f_k is determined by requiring the condition that a fuzzy $2k$-sphere becomes a classical solution of this matrix model. The equation of motion for the matrix model (3.9) is

$$[\Phi_j, [\Phi_i, \Phi_j]] - \frac{i^k}{k!}\lambda^{k-1} f_k \Phi_{i_2}\cdots\Phi_{i_{2k+1}}\epsilon_{ii_2\ldots i_{2k+1}} = 0 . \tag{3.10}$$

We substitute the following ansatz

$$X_i = \lambda\Phi_i = \alpha \hat{G}^{(k)}_i \tag{3.11}$$

into the above equation. The radius of a fuzzy $2k$-sphere is $r^2 = \alpha^2 n(n+2k)$ as in (2.8). We easily find that f_k should be given by

$$f_1 = \frac{4\alpha}{\lambda} , \quad f_2 = \frac{4}{\alpha(n+2)} , \quad f_3 = \frac{3\lambda}{\alpha^3(n+2)(n+4)} , \quad f_4 = \frac{2\lambda^2}{\alpha^5(n+2)(n+4)(n+6)} . \tag{3.12}$$

Note that f_k depends on n when k takes $2, 3, 4$. These f_k are compactly denoted by

$$f_k = -(-i)^k \frac{8k}{C_k} k! \alpha^{-2k+3}\lambda^{k-2} , \tag{3.13}$$

where C_k is defined in (A.7). We can evaluate the value of the potential (3.9) for the fuzzy $2k$-sphere solution (3.11) as

$$V_k = 2k\left(1 - \frac{4}{2k+1}\right) T_0 \frac{\alpha^4}{\lambda^2} n(n+2k) N_k . \tag{3.14}$$

It must be noted that V_1 is negative while $V_k (k = 2, 3, 4)$ are positive.

Let us comment on another classical solution. A set of commuting matrices

$$[\Phi_i, \Phi_j] = 0 \tag{3.15}$$

is also a classical solution. Since Φ_i commute each other, they are simultaneously diagonalized and represent a set of N separated D0-branes. The potential energy (3.9) for this solution is zero. Since V_k in (3.14) is positive when $k = 2, 3, 4$, a fuzzy $2k$-sphere is unstable and is expected to collapse into the solution (3.15). This situation is opposite to the case of fuzzy two-sphere. Since the potential energy of a fuzzy two-sphere is lower than that of the solution (3.15), a fuzzy two-sphere is classically stable [7]. This is one of differences between fuzzy two-sphere and fuzzy $2k$-sphere ($k \neq 2$). Another difference can be found by noticing that f_k depends on n when k takes $2, 3, 4$. Due to the n dependence of f_k, reducible representations cannot be classical solutions. (If we require a reducible representation to be a classical solution of the model, an irreducible representation cannot.) Taking account of these differences, the classical dynamics of higher dimensional fuzzy sphere is completely different from that of fuzzy two-sphere.

A fuzzy $2k$-sphere is obtained by a matrix representation of the size N_k, where N_k is defined in (2.5). The size of matrix N_k is interpreted as the number of D0-branes. Since the value of N_k is restricted, the number of D0-branes is restricted to realize spherical D$2k$-branes from D0-branes. In the next section, N_k is compared to the Chern number c_k on spherical D$2k$-branes.

4. Dual description of fuzzy sphere

In the previous section, we realized a higher dimensional fuzzy sphere as a classical solution of a matrix model of D0-branes. Noncommutative geometry is fully used in this description. The classical solution is really a bound state of D$2k$-branes and D0-branes. On the other hand, we have a dual description. From the viewpoint of a world-volume theory on D$2k$-branes, D0-branes bounded on them are expressed by a nontrivial gauge field configuration. The Chern character of it expresses the charge of D0-branes. In this section, we study a dual description of the higher dimensional fuzzy sphere.[2] This study helps us to understand some unusual aspects of higher dimensional fuzzy spheres. We consider a system of spherical D$2k$-branes and gauge fields on them in a constant R-R $(2k+2)$-form field strength background, and compare the potential energy with (3.14). This calculation is done by using the Born-Infeld action. It is shown that two descriptions coincide in the limit of large-n.

Notation of indices in this section is as follows, $\mu, \nu = 1, 2, \ldots, 2k+1$ and $\alpha, \beta = 1, 2, \ldots, 2k$. a,b are used for the world-volume indices.

4.1 Dual description of fuzzy four-sphere

We first consider a dual description of the fuzzy four-sphere. An instanton solution on a four-sphere space is constructed in [52], and it is given in (B.10);

$$A_\alpha = -\frac{1}{2r(r + x_5)} \eta^i_{\alpha\beta} x_\beta \sigma_i \,, \qquad A_5 = 0 \,, \tag{4.1}$$

[2] A dual description of fuzzy four-sphere and fuzzy six-sphere is discussed in [27, 46] in the context of a fuzzy funnel solution. In this paper, we focus on a relationship between noncommutative geometry and (nonabelian) gauge fields. Some parts of the calculation in these papers overlap with those in our paper.

where $\eta^i_{\alpha\beta}$ is the 't Hooft symbol. The gauge group of the instanton is SU(2). To relate this description with fuzzy sphere, we need to replace σ_i with L_i, where L_i is the spin $n/2$ representation of SU(2). The instanton number is defined as

$$c_2 = -\frac{1}{8\pi^2}\int_{S^4} \mathrm{Tr}\,(F \wedge F)\,. \tag{4.2}$$

The minus sign is our convention. Since the instanton configuration (4.1) has the SO(5) symmetry, we can use the value of F at the north pole to simplify the calculation. From (B.14), the field strength at the north pole ($x_\alpha = 0$, $x_5 = r$) is given by

$$F_{\alpha\beta} = \frac{1}{2r^2}\eta^i_{\alpha\beta} L_i\,, \qquad F_{5\alpha} = 0\,. \tag{4.3}$$

We now calculate the Chern numbers, which provide the charges of lower dimensional D-branes. Since $c_1 = trF = 0$, there are no net D2-brane charge. As can be understood from the fact that L_i form a fuzzy two-sphere, positive charges and negative charges cancel each other. It is locally nonzero as shown in section 5. c_2 is evaluated as follow,

$$\begin{aligned} c_2 &= -\frac{1}{8\pi^2}\frac{1}{4}\Omega_4 r^4\,\mathrm{Tr}\,(\epsilon_{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}) \\ &= \frac{1}{16\pi^2}\Omega_4\,\mathrm{Tr}(L_iL_i) \\ &= \frac{1}{6}n(n+1)(n+2) \equiv \bar{c}_2\,. \end{aligned} \tag{4.4}$$

This corresponds to the number of D0-branes. Therefore a bound state of $\bar{c}_2$ D0-branes[3] and $(n+1)$ D4-branes is realized by introducing the SU(2) gauge field. F_{ab} in (4.3) satisfies the following (anti-)self-dual relation,

$$\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta} = -2F_{\alpha\beta}\,. \tag{4.5}$$

Let us consider the world-volume action for $(n+1)$ D4-branes with the gauge field (4.1). The Born-Infeld action for nonabelian D4-branes is given by

$$S_{BI} = -T_4\int d^{4+1}\sigma\,\mathrm{Str}\,\sqrt{-\det(P[G+\lambda F]_{ab})}\,. \tag{4.6}$$

It is assumed that D4-branes are static and spherical. We fix the position of the D4-branes as $x_i = 0$ ($i = 6, \ldots .9$) and $x_5 = r$, and adopt the static gauge $x_a = \sigma_a$ ($a = 0, 1, \ldots, 4$) around the north pole. The pullback $P[\cdots]$ is calculated as

$$\begin{aligned} P[G+\lambda F]_{ab} &= (G_{\mu\nu}+\lambda F_{\mu\nu})\frac{\partial x^\mu}{\partial\sigma^a}\frac{\partial x^\nu}{\partial\sigma^b} \\ &= G_{ab} + \lambda F_{ab} + G_{i(a}\partial_{b)}x^i + \lambda F_{i[a}\partial_{b]}x^i + (G_{ij}+\lambda F_{ij})\partial_a x^i\partial_b x^j \\ &\to G_{ab} + \lambda F_{ab}\,. \end{aligned} \tag{4.7}$$

[3]Due to the minus sign of c_2, it is really a bound state of anti-D0-branes and D4-branes.

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The determinant is evaluated by using these assumptions as

$$
\begin{aligned}
-\det(G_{ab}+\lambda F_{ab}) &= 1+\frac{\lambda^2}{2}F_{\alpha\beta}F_{\alpha\beta}+\frac{\lambda^4}{64}(F_{\alpha\beta}F_{\gamma\delta}\epsilon^{\alpha\beta\gamma\delta})^2 \\
&= \left(1+\frac{\lambda^2}{4}F_{\alpha\beta}F_{\alpha\beta}\right)^2 ,
\end{aligned} \tag{4.8}
$$

where we have evaluated it around the north pole of a four-sphere by using the rotation symmetry, and a flat metric $G_{ab}=\eta_{ab}$ was used. From the first line to the second line, the self-dual condition (4.5) has been used. In this calculation, we have regard F as commutative in spite of its nonabelian property, which is justified since the determinant is in the symmetrized trace [43]. Then the Born-Infeld action is calculated as

$$
\begin{aligned}
S &= -T_4\int d^{4+1}\sigma\,\mathrm{Str}\,\sqrt{-\det(P[G+\lambda F]_{ab})} \\
&= -T_4\int dt\,\left((n+1)\Omega_4 r^4+\bar{c}_2 4\pi^2\lambda^2\right) \\
&= -\int dt\,\left((n+1)T_4\Omega_4 r^4+\bar{c}_2 T_0\right),
\end{aligned} \tag{4.9}
$$

where $\Omega_4=8\pi^2/3$ is the volume of a four-sphere with unit radius. The first term corresponds to the energy of $(n+1)$ spherical D4-branes with the radius r, and the second term is the energy of $\bar{c}_2$ D0-branes.

We next consider the Chern-Simons coupling. We are now considering a case where only R-R six-form field strength is nonzero. The background R-R six-form field strength (3.7) can provide the following five form potential after we use a gauge choice,

$$
C^{(5)}_{t1234}=\frac{1}{5}f_2x_5\simeq\frac{1}{5}f_2 r\,, \tag{4.10}
$$

where we have evaluated around the north pole $x_5\simeq r$. Then the Chern-Simons term is calculated as

$$
\begin{aligned}
S_{CS} &= \mu_4\,\mathrm{Tr}\int C^{(5)} \\
&= \mu_4\,\mathrm{Tr}\int dt\int d^4x\left(\frac{1}{5}f_2 r\right) \\
&= \mu_4\int dt\left((n+1)\frac{1}{5}f_2\Omega_4 r^5\right).
\end{aligned} \tag{4.11}
$$

By combining the contributions from the Born-Infeld action (4.9) and the Chern-Simons action (4.11), the potential energy for D4-branes is provided by

$$
V(r)=\bar{c}_2T_0+T_4\Omega_4(n+1)\left(r^4-\frac{f_2}{5}r^5\right). \tag{4.12}
$$

We regard this potential as a function of r. It has two extrema, one is given by $r=0$ and another is

$$
r=\frac{4}{f_2}=\alpha(n+2)\equiv r_\star\,. \tag{4.13}
$$

At the first extremum, D4-branes cannot have a nonzero radius and only $\bar{c}_2$ D0-branes can exist. The second one is related to a fuzzy four-sphere solution. $r_\star$ is the radius of the four-sphere, and can be compared to the radius of a fuzzy four sphere which is given in (2.8). Because both becomes αn at large-n, two descriptions provide the same radius at large-n.

From the shape of the potential, the first extremum is a local minimum, while the second one is a local maximum. Therefore the spherical configuration is classically unstable against a small fluctuation. This situation is the same as one encountered in the previous section.

We now calculate the value of the potential for the spherical solution. By substituting $r = r_\star$ into $V(r)$, we have

$$V(r_\star) - \bar{c}_2 T_0 = T_0 \frac{2}{15}\frac{\alpha^4}{\lambda^2}(n+1)(n+2)^4 \simeq T_0 \frac{2}{15}\frac{\alpha^4}{\lambda^2} n^5 \,. \tag{4.14}$$

This should be compared with V_2, which is obtained in (3.14);

$$V_2 = T_0 \frac{2}{15}\frac{\alpha^4}{\lambda^2} n(n+1)(n+2)(n+3)(n+4) \simeq T_0 \frac{2}{15}\frac{\alpha^4}{\lambda^2} n^5 \,. \tag{4.15}$$

These two values agree at large-n. We can also compare the D0-brane charge. In the first description, the size of the matrix represents the D0-brane charge and it is given by N_2. On the other hand, it is $\bar{c}_2$ in the second description. Both behave $n^3/6$ at large-n. We have compared three quantities, the potential energy, the radius and the D0-brane charge. All of them gave the same values in two descriptions when n is large. This result leads to the conclusion that a fuzzy four-sphere is the same object as spherical D4-branes with an SU(2) monopole gauge field at large-n.

Let us comment on the validity of these two descriptions [27, 12]. We first considered the world-volume theory of D0-branes. The assumption $\lambda[\Phi_i, \Phi_j] \ll 1$ was used to derive the low energy effective action of D0-branes (3.2). Since this condition is rewritten as

$$l_{nc}^2 \simeq \frac{r^2}{n} \ll l_s^2 \,, \tag{4.16}$$

the noncommutative scale l_{nc}, which is defined in (2.9), has to be much smaller than the string scale l_s. On the other hand, the computations by using the world-volume action of D2k-branes can be trusted as long as the field strength is slowly varying $\mid l_s \partial F \mid \ll \mid F \mid$. This condition is satisfied when the radius is much larger than the string scale;

$$l_s \ll r \,. \tag{4.17}$$

If r satisfies the following region

$$l_s \ll r \ll \sqrt{n} l_s \,, \tag{4.18}$$

both of (4.16) and (4.17) are satisfied. It is sometimes convenient to rewrite (4.18) as

$$\frac{l_s}{\sqrt{n}} \ll l_{nc} \ll l_s \,. \tag{4.19}$$

We can expect the agreement of two descriptions in a large region by taking a large-n limit. This is the reason why we could obtain the agreement of two descriptions in a large-n limit. We can provide another explanation in section 5.

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4.2 Dual description of fuzzy six-sphere

We next consider a dual description of a fuzzy six-sphere. The idea is basically the same as the case of fuzzy four-sphere.

We begin by introducing an SO(6) gauge field on a six-sphere which is obtained in (B.17):

$$A_\alpha = \frac{-i}{2r(r+x_{2k+1})}\Sigma^N_{\beta\alpha}x_\beta\,, \qquad A_7 = 0 \tag{4.20}$$

where $\Sigma^N_{\alpha\beta} = (\Sigma^N_{ij}, \Sigma^N_{6i}) = (\gamma_{ij}, i\gamma_i) = (\frac{1}{2}[\gamma_i,\gamma_j], i\gamma_i)$, γ_i $(i = 1,\ldots,5)$ is the five-dimensional gamma matrix, whose explicit representation is provided in (B.8). $\Sigma^N_{\alpha\beta}$ transforms in a spinor representation of SO(6). To make a connection to fuzzy sphere, we need to consider a higher dimensional representation of SO(6). Such a representation was already obtained in section 2, and the $k = 2$ case is relevant in the present case. Then we replace the five-dimensional gamma matrices γ_i and γ_{ij} with higher dimensional representations $\hat{G}^{(2)}_i$ and $\hat{G}^{(2)}_{ij}$. The index (2) was added to emphasize that the matrices are related to the $k = 2$ case in section 2. Accordingly $\Sigma^N_{\alpha\beta}$ is replaced with $\hat{\Sigma}^N_{\alpha\beta} = (\hat{\Sigma}^N_{ij}, \hat{\Sigma}^N_{2k,i}) \equiv (\hat{G}^{(2)}_{ij}, i\hat{G}^{(2)}_i)$. Since these matrices are realized by the size N_2 which is defined in (2.5), the gauge field becomes an $N_2 \times N_2$ matrix. A interesting fact is that these matrices construct a fuzzy four-sphere.

Before we begin calculations, let us explain the meaning of the index N in $\hat{\Sigma}^N_{\alpha\beta}$. We consider the N_3-dimensional irreducible representation of SO(7), which is denoted by $\hat{G}^{(3)}_{\mu\nu}$ $(\mu,\nu = 1,\ldots,7)$. As is explained in section 2, it is associated with a fuzzy six-sphere. We now restrict our interest to a subalgebra $\hat{G}^{(3)}_{\alpha\beta}$ $(\alpha,\beta = 1,\ldots,6)$, which forms the SO(6) algebra. $\hat{G}^{(3)}_{\alpha\beta}$ is reducible, and is characterized by the eigenvalues of $\hat{G}^{(3)}_7 = diag(n, n-2,\ldots,-n+2-n)$. $\hat{G}^{(3)}_7$ is a generalized chirality matrix. We consider a subspace which is labelled by $\hat{G}^{(3)}_7 = n1$, which corresponds to the North pole of a fuzzy six-sphere. It can be shown that the size of the unit matrix 1 is N_2. This $\hat{G}^{(3)}_{\alpha\beta}$ which is labelled by $\hat{G}^{(3)}_7 = n1$ is nothing but $\hat{\Sigma}^N_{\alpha\beta}$. We added the index N to clarify a relationship between the fuzzy six-sphere and the SO(6) nonabelian gauge field.

The field strength at the north pole ($x_\alpha = 0$, $x_7 = r$), which is calculated in (B.18), is

$$F_{\alpha\beta} = -\frac{i}{2r^2}\hat{\Sigma}^N_{\alpha\beta}\,, \qquad F_{7\alpha} = 0\,. \tag{4.21}$$

The calculation of the Chern numbers is easily done by making full use of some formulae in appendix A. The first and second Chern numbers are calculated as $c_1 = \mathrm{Tr}\,F \simeq \mathrm{Tr}\,\Sigma_{\alpha\beta} = 0$ and

$$\begin{aligned} c_2 &\sim \mathrm{Tr}\,(F\wedge F) \\ &\sim \epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4}\,\mathrm{Tr}\,(F_{\alpha_1\alpha_2}F_{\alpha_3\alpha_4}) \\ &\sim \epsilon_{i_1i_2i_3i_4}\,\mathrm{Tr}\left(\hat{G}^{(2)}_{i_1i_2}\hat{G}^{(2)}_{i_3i_4}\right) + \epsilon_{i_1i_2i_3}\,\mathrm{Tr}\left(\hat{G}^{(2)}_{i_1i_2}\hat{G}^{(2)}_{i_3}\right) \\ &\sim \mathrm{Tr}\left(\hat{G}^{(2)}_i\right) + \mathrm{Tr}\left(\hat{G}^{(2)}_{ij}\right) = 0\,. \end{aligned} \tag{4.22}$$

The net charge of D2-branes and that of D4-branes vanish. This is equivalent to the fact that the net charge of a fuzzy two-sphere and that of a fuzzy four-sphere are zero. The third Chern number is

$$\begin{aligned}
c_3 &= \frac{1}{48\pi^3}\int_{S^6} \mathrm{Tr}\,(F\wedge F\wedge F) \\
&= \frac{1}{48\pi^3}\frac{1}{8}\Omega_6 r^6 \sum_{\alpha_i=1}^{6} \mathrm{Tr}\,(F_{\alpha_1\alpha_2}F_{\alpha_3\alpha_4}F_{\alpha_5\alpha_6})\,\epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6} \\
&= \frac{1}{512\pi^3}\Omega_6 \sum_{i_i=1}^{5} \mathrm{Tr}\left(\hat{G}^{(2)}_{i_1}\hat{G}^{(2)}_{i_2}\hat{G}^{(2)}_{i_3}\hat{G}^{(2)}_{i_4}\hat{G}^{(2)}_{i_5}\right)\epsilon_{i_1i_2i_3i_4i_5} \\
&= \frac{1}{512\pi^3}\Omega_6(8n+16)\,\mathrm{Tr}(\hat{G}^{(2)}_i\hat{G}^{(2)}_i) \\
&= \frac{1}{360}n(n+1)(n+2)^2(n+3)(n+4) \equiv \bar{c}_3\,.
\end{aligned} \tag{4.23}$$

$\Omega_6 = 16\pi^3/15$ is the volume of a six-sphere with unit radius. This corresponds to the number of D0-branes, and can be compared with N_3 in (2.5). We easily find that these coincide at large-n. The use of the nonabelian gauge field (4.20) allows us to construct a bound state of $\bar{c}_3$ D0-branes and N_2 D6-branes. We note that (4.21) satisfies

$$\epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6}F^{\alpha_3\alpha_4}F^{\alpha_5\alpha_6} = \frac{4}{r^2}(n+2)F_{\alpha_1\alpha_2}\,, \tag{4.24}$$

which is a natural generalization of a self-dual equation of the instanton.

We now consider a world-volume action for N_2 $D6$-branes with the gauge field (4.21). The Born-Infeld action is

$$S_{BI} = -T_6\int d^{6+1}\sigma\,\mathrm{Str}\sqrt{-\det(P[G+\lambda F]_{ab})}\,. \tag{4.25}$$

The determinant is evaluated as follows,

$$\begin{aligned}
-\det(G_{ab}+\lambda F_{ab}) &= 1+\frac{\lambda^2}{2}F_{\alpha\beta}F_{\alpha\beta}+\frac{\lambda^4}{128}(F_{\alpha_1\alpha_2}F_{\alpha_3\alpha_4}\epsilon^{\alpha\beta\alpha_1\alpha_2\alpha_3\alpha_4})\times \\
&\quad\times(F_{\beta_1\beta_2}F_{\beta_3\beta_4}\epsilon^{\alpha\beta\beta_1\beta_2\beta_3\beta_4})+\frac{\lambda^6}{2304}(\epsilon^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6}F_{\alpha_1\alpha_2}F_{\alpha_3\alpha_4}F_{\alpha_5\alpha_6})^2 \\
&= 1+\frac{3}{4}\left(\frac{\lambda}{r^2}\right)^2 n(n+6)+\frac{3}{16}\left(\frac{\lambda}{r^2}\right)^4 n(n+2)^2(n+6)+ \\
&\quad+\frac{1}{64}\left(\frac{\lambda}{r^2}\right)^6 n^2(n+2)^2(n+4)^2 \\
&\simeq 1+\frac{3}{4}\left(\frac{\lambda n}{r^2}\right)^2+\frac{3}{16}\left(\frac{\lambda n}{r^2}\right)^4+\frac{1}{64}\left(\frac{\lambda n}{r^2}\right)^6,
\end{aligned} \tag{4.26}$$

where we have assumed a large-n to obtain the last expression. We define $y \equiv r^2/\lambda n$. The Born-Infeld action becomes

$$\frac{S_{BI}}{\int dt} = -T_6\Omega_6 r^6 N_2\sqrt{1+\frac{3}{4y^2}+\frac{3}{16y^4}+\frac{1}{64y^6}}$$

$$
\begin{aligned}
&= -T_6\Omega_6 r^6 N_2 \frac{1}{8y^3}\sqrt{1+12y^2+48y^4+64y^6} \\
&\simeq -T_6\Omega_6 r^6 N_2 \left(\frac{1}{8y^3}+\frac{3}{4y}+O(y)\right) \\
&\simeq -\bar{c}_3 T_0 - \frac{3}{4}T_6 N_2 \Omega_6 r^4 \lambda n\,.
\end{aligned} \tag{4.27}
$$

To arrive the last expression, we have expanded the square root by assuming the condition $y \ll 1$. The use of this assumption is valid since it satisfies (4.18). We next consider the coupling of D6-branes to the external R-R field. The constant R-R eight-form field strength background (3.7) provides the following R-R seven-form field in a certain gauge,

$$
C^{(7)}_{t123456} = \frac{1}{7}f_3 x_7 \simeq \frac{1}{7}f_3 r\,. \tag{4.28}
$$

The Chern-Simons term is calculated as

$$
S_{CS} = \mu_6 \int dt \left(N_2 \frac{f_3}{7}\Omega_6 r^7\right), \tag{4.29}
$$

and the potential for the D6-branes is provided by

$$
V(r) = \bar{c}_3 T_0 + T_6\Omega_6 N_2 \left(\frac{3}{4}\lambda n r^4 - \frac{f_3}{7}r^7\right). \tag{4.30}
$$

The first term represents the rest energy of $\bar{c}_3$ D0-branes. We search for extrema of this potential by regarding it as a function of r. We can find two extrema, one is trivial extremum $r = 0$, and another is

$$
r = \sqrt[3]{\frac{3\lambda n}{f_3}} = \sqrt[3]{\alpha^3 n(n+2)(n+4)} \simeq \alpha n \equiv r_\star\,. \tag{4.31}
$$

This extremum corresponds to a fuzzy six-sphere solution in the matrix model of D0-branes. The radius of this spherical solution agrees with that of the fuzzy six sphere (2.8) at large-n. The potential value for this extremum is found to be

$$
V(r_\star) - \bar{c}_3 T_0 \simeq T_0 \frac{1}{140}\frac{\alpha^4}{\lambda^2}n^8\,. \tag{4.32}
$$

We recall the D0-brane calculation in the previous section;

$$
\begin{aligned}
V_3 &= T_0 \frac{1}{140}\frac{\alpha^4}{\lambda^2} n(n+1)(n+2)(n+3)^2(n+4)(n+5)(n+6) \\
&\simeq T_0 \frac{1}{140}\frac{\alpha^4}{\lambda^2} n^8\,.
\end{aligned} \tag{4.33}
$$

$V(r_\star)$ and V_3 give the same value including the coefficient when n is large. These results manifest the fact that a fuzzy six-sphere of the matrix size N_3 is the same object as N_2 spherical D6-branes with a nonabelian SO(6) gauge field.

4.3 Dual description of fuzzy eight-sphere

It is straightforward to generalize the calculations in the previous two cases to a dual description of a fuzzy eight-sphere. The detailed calculations are almost analogous to the previous cases. We start with an SO(8) monopole field on an eight-sphere;

$$A_\alpha = \frac{-i}{2r(r+x_{2k+1})}\Sigma^N_{\beta\alpha}x_\beta\,, \qquad A_9 = 0 \tag{4.34}$$

where $\Sigma^N_{\alpha\beta} = (\Sigma^N_{ij}, \Sigma^N_{8i}) = (\gamma_{ij}, i\gamma_i)$, and γ_i $(i = 1, \ldots, 7)$ is the seven-dimensional gamma matrix. The gauge field belongs to a spinor representation of SO(8). To construct a higher dimensional representation, we replace $\Sigma^N_{\alpha\beta}$ with $\hat{\Sigma}^N_{\alpha\beta} \equiv (\hat{G}^{(3)}_{ij}, i\hat{G}^{(3)}_i)$. The index (3) means that these are the matrices of the $k = 3$ case in section 2. These are realized by an $N_3 \times N_3$ matrices, where N_3 is presented in (2.5). Note that $\hat{G}^{(3)}_{ij}$ and $\hat{G}^{(3)}_i$ construct a fuzzy six-sphere.

The gauge field strength is calculated in (B.18), and it becomes the following form at the north pole ($x_\alpha = 0$, $x_9 = r$);

$$F_{\alpha\beta} = -\frac{i}{2r^2}\hat{\Sigma}^N_{\alpha\beta}\,, \qquad F_{9\alpha} = 0\,. \tag{4.35}$$

The vanishing of the Chern numbers $c_1 = c_2 = c_3 = 0$ is shown by the analogous calculations to (4.22) with the help of some properties of $\hat{G}^{(3)}_i$ and $\hat{G}^{(3)}_{ij}$. Therefore the net charge of D2k'-brane ($k' = 1, 2, 3$), which forms a fuzzy $2k'$-sphere, is zero. The forth Chern number is nonzero;

$$\begin{aligned}
c_4 &= -\frac{1}{4!(2\pi)^4}\int_{S^8} \mathrm{Tr}\,(F\wedge F\wedge F\wedge F) \\
&= -\frac{1}{4!(2\pi)^4}\frac{1}{2^4}\Omega_8 r^8 \sum_{\alpha_i=1}^{8} \mathrm{Tr}\,(F_{\alpha_1\alpha_2}F_{\alpha_3\alpha_4}F_{\alpha_5\alpha_6}F_{\alpha_7\alpha_8})\,\epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7\alpha_8} \\
&= \frac{i}{4!(2\pi)^4}\frac{1}{2^5}\Omega_8 \sum_{i_i=1}^{7} \mathrm{Tr}\left(\hat{G}^{(3)}_{i_1}\hat{G}^{(3)}_{i_2}\hat{G}^{(3)}_{i_3}\hat{G}^{(3)}_{i_4}\hat{G}^{(3)}_{i_5}\hat{G}^{(3)}_{i_6}\hat{G}^{(3)}_{i_7}\right)\epsilon_{i_1i_2i_3i_4i_5i_6i_7} \\
&= \frac{1}{302400}n(n+1)(n+2)^2(n+3)^2(n+4)^2(n+5)(n+6) \equiv \bar{c}_4\,.
\end{aligned} \tag{4.36}$$

$\Omega_8 = 32\pi^4/105$ is the volume of an eight-sphere with unit radius. This represents the number of D0-branes. We can confirm the agreement with N_4 at large-n. We note that (4.35) satisfies

$$\epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7\alpha_8}F^{\alpha_3\alpha_4}F^{\alpha_5\alpha_6}F^{\alpha_7\alpha_8} = -\frac{12}{r^4}(n+2)(n+4)F_{\alpha_1\alpha_2}\,. \tag{4.37}$$

Let us next consider a world-volume theory on D8-branes. The dynamics of D8-branes with a gauge field is described by the Born-Infeld action;

$$S_{BI} = -T_8\int d^{8+1}\sigma\,\mathrm{Str}\,\sqrt{-\det(P[G+\lambda F]_{ab})}\,. \tag{4.38}$$

The determinant around the north pole becomes

$$\begin{aligned}
-\det(G_{ab}+\lambda F_{ab}) \;=\; & 1+\frac{\lambda^2}{2}F_{\alpha\beta}F_{\alpha\beta} \\
& +\frac{\lambda^4}{1536}(F_{\alpha_1\alpha_2}F_{\alpha_3\alpha_4}\epsilon^{\alpha\beta\gamma\delta\alpha_1\alpha_2\alpha_3\alpha_4})(F_{\beta_1\beta_2}F_{\beta_3\beta_4}\epsilon^{\alpha\beta\gamma\delta\beta_1\beta_2\beta_3\beta_4}) \\
& +\frac{\lambda^6}{4608}\left(\epsilon^{\alpha\beta\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6}F_{\alpha_1\alpha_2}F_{\alpha_3\alpha_4}F_{\alpha_5\alpha_6}\right)\times \\
& \times\left(\epsilon^{\alpha\beta\beta_1\beta_2\beta_3\beta_4\beta_5\beta_6}F_{\beta_1\beta_2}F_{\beta_3\beta_4}F_{\beta_5\beta_6}\right) \\
& +\frac{\lambda^8}{147456}(\epsilon^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7\alpha_8}F_{\alpha_1\alpha_2}F_{\alpha_3\alpha_4}F_{\alpha_5\alpha_6}F_{\alpha_7\alpha_8})^2\,.
\end{aligned} \tag{4.39}$$

By assuming the condition $r^2 \ll \lambda n$, and using (4.37) and some formulae in appendix A, the resulting action is

$$S_{BI} \simeq \int dt\left(-\bar{c}_4 T_0 - \frac{1}{2}T_8 N_3 \Omega_8 r^4 \lambda^2 n^2\right). \tag{4.40}$$

The Chern-Simons term is also calculated as

$$S_{CS} = \mu_8 \int dt\left(N_3 \frac{f_4}{9}\Omega_8 r^9\right), \tag{4.41}$$

where we have used the fact that the R-R field is given by the following form in a certain gauge,

$$C^{(9)}_{t12345678} = \frac{1}{9}f_4 x_9 \simeq \frac{1}{9}f_4 r\,. \tag{4.42}$$

Therefore the dynamics of static D8-branes with the gauge field in a constant R-R field strength background is described by

$$V(r) = \bar{c}_4 T_0 + T_8 \Omega_8 N_3\left(\frac{1}{2}r^4\lambda^2 n^2 - \frac{f_4}{9}r^9\right). \tag{4.43}$$

This potential indicates two extrema, $r=0$ and

$$r = \sqrt[5]{\frac{2\lambda^2 n^2}{f_4}} \simeq \alpha n \equiv r_\star\,. \tag{4.44}$$

The second one can be compared with the radius of a fuzzy eight-sphere. $r_\star$ surely agrees with the radius of a fuzzy eight-sphere at large-n. The potential value for $r = r_\star$ is

$$V(r_\star) - \bar{c}_4 T_0 \simeq T_0 \frac{1}{68040}\frac{\alpha^4}{\lambda^2}n^{12}\,. \tag{4.45}$$

This presents the same large-n behavior as V_4 including the numerical coefficient. One can state that a fuzzy eight-sphere is the same object as D8-branes with the nonabelian gauge field (4.34) when n is large.

5. Noncommutativity and nonabelian gauge field

In the previous sections, we considered two world-volume theories. Since they provide the same values for various quantities at large-n, they are supposed to be the same thing. In this section, we discuss a concrete correspondence of them by noticing a relationship between the noncommutativity of fuzzy sphere and nonabelian gauge fields. The results presented in the previous section confirmed that a fuzzy $2k$-sphere is dual to D$2k$-branes with an SO$(2k)$ nonabelian gauge field. An interesting fact is that the nonabelian gauge field on D$2k$-branes is expressed by matrices which are related to the fuzzy $(2k-2)$-sphere. We recall the gauge field strength evaluated at the north pole ($x_\alpha = 0$, $x_{2k+1} = r$);

$$F_{\alpha\beta} = -\frac{i}{2r^2}\hat{\Sigma}^N_{\alpha\beta}\,, \tag{5.1}$$

where $\hat{\Sigma}^N_{\alpha\beta} = (\hat{\Sigma}^N_{ij}, \hat{\Sigma}^N_{2k,i}) = (\hat{G}^{(k-1)}_{ij}, i\hat{G}^{(k-1)}_i)$ $(i = 1, \ldots, 2k-1)$. The matrices $\hat{G}^{(k-1)}_{ij}$ and $\hat{G}^{(k-1)}_i$ are realized by the N_{k-1}-dimensional irreducible representation of SO$(2k-1)$, forming the fuzzy $(2k-2)$-sphere algebra.

A commutation relation of coordinates of a $2k$-fuzzy sphere is given by the first equation in (2.6). This relation reduces to the following relation at the north pole of a fuzzy $2k$-sphere ($\hat{G}_{2k+1} = n$),

$$[\hat{G}^N_\alpha, \hat{G}^N_\beta] = 2\hat{G}^N_{\alpha\beta}\,, \tag{5.2}$$

where $\alpha, \beta = 1, \ldots, 2k$. We should not confuse two kinds of the north pole. As is explained in 4.2, $\hat{G}^N_{\alpha\beta}$ is the same matrix as $\Sigma^N_{\alpha\beta}$. Both are realized by an N_{k-1}-dimensional representation of SO$(2k)$. Since the two descriptions agree in the large-n limit, we may combine two relations (5.1) and (5.2) only in the large-n limit. Therefore we are led to the following noncommutative relation for coordinates of a sphere,

$$[X_\alpha, X_\beta] = 4i\alpha^2 r^2 F_{\alpha\beta}\,, \tag{5.3}$$

where $X_\alpha = \alpha\hat{G}^N_\alpha$. We have identified the noncommutative coordinates with commutative coordinates in the dual description, and the noncommutativity of coordinates has been given by the field strength. This relation suggests the identification of the noncommutative coordinate with the covariant derivative under the SO$(2k)$ monopole background. This is also expected from the result in [22], where it was shown that an adjoint action of $\hat{G}_\mu$ is mapped to the covariant derivative under the monopole background for the $k = 2$ case.[4]

The commutation relation (5.3) is valid at the north pole. It is natural to expect that this relation exists without restricting to the north pole because a (fuzzy) $2k$-sphere has the SO$(2k+1)$ symmetry. Therefore we suppose

$$[X_\mu, X_\nu] = 2\alpha^2\hat{G}_{\mu\nu} = 4i\alpha^2 r^2 F_{\mu\nu}\,, \tag{5.4}$$

where the size of the matrices is N_k and $\mu, \nu = 1, \ldots, 2k+1$. $F_{\mu\nu}$ has been expressed by the N_k-dimensional irreducible representation of SO$(2k+1)$. This relation is interesting in the

[4] Functions on fuzzy spheres are expanded by an extended version of spherical harmonics [19]. The spectrum of $ad(G)^2$ was calculated by acting on the spherical harmonics, and compared it to the hamiltonian in the four-dimensional quantum Hall system, which is the square of the covariant derivative.

following sense. In the fuzzy sphere algebra (2.4) and (2.6), $\hat{G}_{\mu\nu}$ and $\hat{G}_\mu$ are treated on the same footing. If we regard $\hat{G}_\mu$ as coordinates of a fuzzy sphere, $\hat{G}_{\mu\nu}$ acts on them as the SO$(2k+1)$ rotation generator (we may regard $\hat{G}_{\mu\nu}$ as coordinates of an extra space.). On the other hand, the role of coordinates and that of a field strength in the dual description are clearly different. The relation (5.4) suggests that the internal degrees of freedom $F_{\mu\nu}$ should be identified with the rotation generator. This is one of the characteristic aspects of noncommutative geometry. Such identification can be seen as the lowest Landau level physics in a higher dimensional quantum Hall system [29, 30, 32].[5] It is obtained by considering a motion of electrons in a monopole gauge field background [52]. Considering the analogy to the quantum Hall system can provide an intuitive explanation for the agreement of two descriptions at large-n. In this system, the angular momentum operator $\Lambda_{\mu\nu} = -i(x_\mu D_\nu - x_\nu D_\mu)$ and the field strength (B.18) have the following relation;

$$G_{\mu\nu} = \Lambda_{\mu\nu} + 2ir^2 F_{\mu\nu} \,. \tag{5.5}$$

$\Lambda_{\mu\nu}$ does not satisfy the SO$(2k+1)$ algebra due to the existence of the monopole background, and it is $G_{\mu\nu}$ that satisfies the SO$(2k+1)$ algebra. The angular momentum generated by $\Lambda_{\mu\nu}$ characterizes the Landau level, and therefore the representation of $G_{\mu\nu}$ depends on the Landau level. The restriction to the lowest Landau level is achieved by $\Lambda_{\mu\nu} \simeq 0$. Since the magnitude of the field strength F in (5.1) is given by $O(n/r^2)$, the contribution of the second term in the right hand side of (5.5) becomes large compared to the first term in a large-n limit. Therefore the field strength $F_{\mu\nu}$ is identified with the rotation generator $G_{\mu\nu}$ in the lowest Landau level. It can be also shown that $G_{\mu\nu}$ is given by the spinor representation of SO$(2k+1)$ [29, 30, 32]. This limit is just the strong magnetic field limit. As is well known in the two-dimensional quantum Hall system, guiding center coordinates are identified with coordinates of electrons in the strong magnetic field limit. Accordingly coordinates of electrons are described by noncommutative geometry. Fuzzy spheres are actually realized in the higher dimensional quantum Hall system after we take the lowest Landau level limit. This is an intuitive explanation for the agreement of two descriptions in the large-n limit.

In the previous paragraph, we have discussed a large-n limit. We have two parameters r and l_{nc}, and the ratio between them is $r/l_{nc} \simeq \sqrt{n}$. Therefore there are two kinds of large-n limit . One is realized by taking a large r limit with keeping the noncommutative scale l_{nc}, the other by taking a small l_{nc} limit with keeping the radius r. Note that the radius r is not a physical radius of the sphere. These two limits have the following meanings. If we see a fuzzy sphere from a short distance, a noncommutative structure can be seen. On the other hand, if we see from a long distance, a fuzzy sphere looks like a commutative sphere. These parameters have to satisfy (4.18) or (4.19). In the first limit, we should consider (4.19). We need to take a large-n limit with keeping l_{nc} small compared to the string scale. In the second limit, we need to take a large-n limit with keeping r large compared to the string scale. The first limit is the same as one which is used in [47].

[5] Some works investigating higher dimensional quantum Hall system have also been reported in [53, 54, 55, 56, 31].

We shall now look more carefully into the equation (5.4). In (5.4), the field strength was regarded to be given by $\hat{G}_{\mu\nu}$ as

$$\hat{F}_{\mu\nu} = \frac{1}{2ir^2}\hat{G}_{\mu\nu} \,, \tag{5.6}$$

where $r^2 = \alpha^2 n(n+2k)$. The field strength on a (commutative) $2k$-sphere is originally given in (B.18). The equation (5.6) manifests that the field strength is completely expressed by a matrix. In other words, a noncommutative description of (B.18) is (5.6). The coordinate dependence has disappeared, and the information of the coordinate is given by the eigenvalue of the matrix. The index of the matrix represents not only the color space but also the coordinate on the sphere. The noncommutativity of the coordinate and that of the nonabelian gauge field are mixed. We can no longer distinguish them. Noncommutative geometry mixes the two kinds of space in an interesting way. These facts can be summarized in the following sentences. A fuzzy $2k$-sphere is provided by an $N_k \times N_k$ matrix. This means that N_k quanta (or D-particles) form a fuzzy $2k$-sphere. The space of a nonabelian gauge field is formed by N_{k-1} quanta, and N_k/N_{k-1} represents the number of points on the sphere. The matrix space of a fuzzy $2k$-sphere has a structure such as locally $S^{2k}\times$ fuzzy S^{2k-2}, where the fuzzy S^{2k-2} construct a nonabelian gauge field. This structure was suggested in [20, 21] from the algebraic point of view.

In the latter part of this section, we consider a large-n limit which relates a fuzzy sphere with a fuzzy plane. Since the radius r is related to n, we can make r large by taking a large-n. In this limit, a fuzzy sphere looks like a fuzzy plane and we can formally obtain a fuzzy plane. Such a large-n limit for the fuzzy sphere algebra was discussed in [21]. The fuzzy sphere algebra basically reduces to some sets of the Heisenberg algebra ($\sim$ two-dimensional noncommutative plane). Since the fuzzy $2k$-sphere is a $k(k+1)$-dimensional space, $k(k+1)/2$ sets of the Heisenberg algebra are obtained. We now confirm that the field strength reduces to abelian (or commutative) because a two-dimensional noncommutative plane is realized by introducing an abelian gauge field. Since the field strength is given in terms of coordinates of (lower) dimensional fuzzy sphere as in (5.1), we can use the calculation of [21]. We show only results. For the $k=2$ case, the field strength reduces to

$$F_{12} = -F_{34} \simeq \frac{n}{2r^2}1 = \frac{1}{2l_{nc}^2}1 \,, \tag{5.7}$$

and other components are zero. We thus obtained two U(1) gauge fields. Nonzero components of the field strength for the $k=3$ and $k=4$ cases can also be shown as

$$F_{12} = -F_{34} = -F_{56} \simeq \frac{n}{2r^2}1 = \frac{1}{2l_{nc}^2}1 \tag{5.8}$$

and

$$F_{12} = -F_{34} = -F_{56} = -F_{87} \simeq \frac{n}{2r^2}1 = \frac{1}{2l_{nc}^2}1 \tag{5.9}$$

respectively. As expected from the correspondence between a noncommutative plane and an abelian gauge field, the nonabelian gauge fields have reduced to some abelian ones. The charges of lower dimensional branes no longer vanish after we take this large-n limit.

6. Summary and discussions

To describe higher dimensional D-branes by using lower dimensional D-branes, we need noncommutative geometry. This description is closely related to the matrix model description of D-branes. It is known that such a higher dimensional D-brane is not just pure a D-brane but a bound state of a higher dimensional D-brane and lower dimensional D-branes. Lower dimensional D-branes bounded on them are expressed by a gauge field configuration with nonvanishing Chern characters. In this sense, gauge field configurations on D-branes are dual to noncommutative geometry. To understand the dual description is directly connected with to understand noncommutative geometry. We expect that such studies help us to know the role of noncommutative geometry in string theory.

To consider such relationships not only for flat D-branes but also for curved ones is an important subject. A fuzzy sphere is used in matrix models to construct a spherical geometry. We can interpret it as a bound state of spherical D-branes and D0-branes. D0-branes on higher dimensional spherical D-branes are regarded by a nonabelian gauge configuration. Higher dimensional fuzzy spheres have some unusual features. One of them is that the number of the dimension of fuzzy spheres is different from that of usual spheres. The dimension of a fuzzy $2k$-sphere is $k(k+1)$. $k(k-1)$ dimensions are clearly extra compared to an usual $2k$-sphere. The role of them for constructing noncommutative geometry is essential. The origin of them has been considered as the use of nonabelian gauge fields. In this paper, we considered two description for a bound state of D0-branes and D$2k$-branes in a constant R-R $(2k+2)$-form field strength background. The first is the D0-brane description, and the second is the D$2k$-brane description. A fuzzy $2k$-sphere appears as a classical solution of a matrix model of D0-branes. A dual description of this is obtained by introducing nonabelian gauge fields. We compared some quantities such as the values of the potential, the radius of sphere and lower dimensional brane charges for these two descriptions. These two descriptions provide different results because each description can be trusted in different parameter regions. Taking a large-n limit leads to the agreement of various quantities including the coefficients. We provided an explanation for the large-n limit by considering the analogy to a quantum Hall system. The large-n limit can be interpreted as the lowest Landau level condition. We finally arrived at the conclusion that a fuzzy $2k$-sphere is the same object as N_{k-1} D$2k$-branes with an SO$(2k)$ gauge field in the limit of large-n. When n is large, we can relate commutative variables with noncommutative variables. Not only the coordinates on spherical D-branes but also a nonabelian gauge field strength are expressed by noncommuting matrices.

The fuzzy sphere algebra is composed of two kinds of matrix, $\hat{G}_\mu$ and $\hat{G}_{\mu\nu}$. Although both form the noncommutative algebra (2.4) and (2.6) and are treated on the same footing, the origin of noncommutativity of them is different. Noncommutativity of $\hat{G}_\mu$ is due to the existence of a magnetic field. On the other hand, that of $\hat{G}_{\mu\nu}$ comes from the nonabelian property of the field strength.

Investigating the dynamics of curved D-branes is an important subject. The fuzzy two-sphere shows an interesting classical dynamics. Reducible representations of SU(2) including separated D0-branes are unstable, condensing to an irreducible representation as a stable state [44].[6] It is, however, not expected to see a similar phenomenon in the case of higher dimensional fuzzy spheres. A big difference which distinguishes higher dimensional fuzzy sphere from fuzzy two-sphere is that N_1 can take any integers while $N_k (k \neq 1)$ cannot. The fact that the size of the matrix is limited is due to the use of higher dimensional algebra.[7] This is related to the fact that the coefficient of the coupling to the R-R field strength depends on n (see (3.12)). These facts which cannot be seen in the case of fuzzy two-sphere would restrict the dynamics of higher dimensional noncommutative branes. Another difference is found by seeing the equation (3.14), which shows that classical energy of higher dimensional spheres is higher than that of D0-branes. Accordingly higher dimensional fuzzy spheres have a classical instability. Taking account of these facts, to study the dynamics of higher dimensional fuzzy spheres including quantum corrections [48, 49, 50] is an interesting future subject.

It is also interesting to investigate a dual description for another noncommutative curved brane. It would be worth while examining a noncommutative CP^2, which is realized by the SU(3) algebra. Some extra dimensions also exist in noncommutative CP^2, which is studied from the algebraic point of view in [45]. Exactly speaking, the existence of them depends on the representation of SU(3). The choice of the representation is related to the choice of the gauge group in a dual description. The construction of the quantum Hall system on CP^2 [51] suggests this relationship.

Acknowledgments

I would like to express my gratitude to K. Hasebe for helpful discussions.

A. Some formulae of fuzzy sphere

In this appendix, we summarize several formulae involving $\hat{G}_\mu$ and $\hat{G}_{\mu\nu}$ in diverse dimensions. The dimension N_k is given by

$$\begin{aligned} N_1 &= n+1\,, \qquad N_2 = \frac{1}{6}(n+1)(n+2)(n+3)\,, \\ N_3 &= \frac{1}{360}(n+1)(n+2)(n+3)^2(n+4)(n+5)\,, \\ N_4 &= \frac{1}{302400}(n+1)(n+2)(n+3)^2(n+4)^2(n+5)^2(n+6)(n+7)\,, \end{aligned} \tag{A.1}$$

where n is a positive integer. We have the following relations

$$\hat{G}_\mu \hat{G}_\mu = n(n+2k) \tag{A.2}$$

and

$$\hat{G}_{\mu\nu} \hat{G}_{\nu\mu} = 2kn(n+2k)\,. \tag{A.3}$$

[6] This phenomenon is investigated from the viewpoint of the tachyon condensation in [45].

[7] This is also true for a noncommutative CP^2, which is constructed from the SU(3) algebra.

The following relations are also satisfied

$$\hat{G}_{\mu\nu}\hat{G}_{\nu} = 2k\hat{G}_{\mu} \tag{A.4}$$

and

$$\hat{G}_{\mu\nu}\hat{G}_{\nu\lambda} = n(n+2k)\delta_{\mu\lambda} + (k-1)\hat{G}_{\mu}\hat{G}_{\lambda} - k\hat{G}_{\lambda}\hat{G}_{\mu}\,. \tag{A.5}$$

G_μ satisfy the following relation

$$\epsilon^{\mu_1\cdots\mu_{2k}\mu_{2k+1}}\hat{G}_{\mu_1}\cdots\hat{G}_{\mu_{2k}} = C_k\hat{G}_{\mu_{2k+1}} \tag{A.6}$$

where $\epsilon^{\mu_1\cdots\mu_{2k}\mu_{2k+1}}$ is the SO$(2k+1)$ invariant tensor. C_k is a constant which depends on n,

$$C_1=2i\,,\qquad C_2=8(n+2)\,,\qquad C_3=-48i(n+2)(n+4)\,,\qquad C_4=-384(n+2)(n+4)(n+6)\,. \tag{A.7}$$

The details of this calculation are given in [23]. By multiplying the equation (A.6) by $\hat{G}_{\mu_{2k+1}}$, we have

$$\begin{aligned}\epsilon^{\mu_1\cdots\mu_{2k}\mu_{2k+1}}\hat{G}_{\mu_1}\cdots\hat{G}_{\mu_{2k}}\hat{G}_{\mu_{2k+1}} &= C_k n(n+2k)\\ &= \frac{i}{2(k+1)}nC_{k+1}\,,\end{aligned} \tag{A.8}$$

where we have used the following relation which is found from (A.7),

$$C_k = -i2k(n+2k-2)C_{k-1}\,. \tag{A.9}$$

B. Hopf map and Berry phase

In this section, we review the construction of the monopole gauge field on an even-dimensional sphere [52, 58] by using the Hopf map [59, 29, 30, 32].

We now consider the first Hopf map. It is known as a map from S^3 to S^2, and naturally introduces an U(1) bundle on S^2. We prepare the following projection operator,

$$P = \frac{1}{2}(1_2 + n_\mu\sigma_\mu) = \begin{pmatrix} 1+n_3 & n_1-in_2\\ n_1+in_2 & 1-n_3\end{pmatrix}, \tag{B.1}$$

which satisfies $P^2 = P$. $x_\mu = rn_\mu$ is a coordinate of S^2. The eigenstate of P is given by

$$|v_N\rangle = \frac{1}{N_N}P\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{N_1}\begin{pmatrix} r+x_3\\ x_1+ix_2\end{pmatrix}, \tag{B.2}$$

where $N_N = \sqrt{2r(r+x_3)}$ is a normalization factor, which ensures $\langle v|v\rangle = 1$. The Berry phase is defined [59] as

$$\gamma_N = -i\int_0^t d\tau\langle v(\tau)|\frac{d}{d\tau}|v(\tau)\rangle = -i\int_0^t\langle v|d|v\rangle = \int_0^t A_N^\mu dx_\mu\,. \tag{B.3}$$

A_μ^N is the Dirac monopole:

$$A_N = A_N^\mu dx_\mu = \frac{1}{2r(r+x_3)}(x_1 dx_2 - x_2 dx_1) = \frac{1}{2r(r+x_3)}\epsilon_{ab} x_a dx_b\,, \tag{B.4}$$

which is singular at $x_3 = -r$. A monopole solution which is singular at $x_3 = r$ is obtained by replacing $|v_N\rangle$ with

$$|v_S\rangle = \frac{1}{N_S} P \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{B.5}$$

where $N_S = \sqrt{2r(r-x_3)}$. The field strength of the Dirac monopole is calculated as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \frac{1}{2r^3}\epsilon_{\mu\nu\rho} x^\rho\,. \tag{B.6}$$

We next consider the second Hopf map: $S^7 \to S^4$. This gives an SU(2) bundle on S^4. The construction can be done in the same way as in the first Hopf map. The projection operator we need in this case is

$$P = \frac{1}{2}(1_4 + n_\mu \Gamma_\mu) = \begin{pmatrix} 1+n_5 & n_4 - i n_i \sigma_i \\ n_4 + i n_i \sigma_i & 1 - n_5 \end{pmatrix}. \tag{B.7}$$

Our notation of the five-dimensional gamma matrix is

$$\begin{aligned} \Gamma_i &= \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \qquad (i = 1,2,3) \\ \Gamma_4 &= \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \qquad \Gamma_5 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \end{aligned} \tag{B.8}$$

where σ_i is the Pauli matrices. They satisfy the Clifford algebra

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu\nu}\,. \tag{B.9}$$

The SU(2) gauge potential is obtained by calculating the Berry phase,

$$\begin{aligned} A_N = A_N^\mu dx_\mu &= \frac{1}{2r(r+x_5)}(\sigma_{ij} x_i dx_j + \sigma_i x_4 dx_i - \sigma_i x_i dx_4) \\ &= -\frac{1}{2r(r+x_5)}\eta^i_{\alpha\beta} x_\beta \sigma_i dx_\alpha\,, \end{aligned} \tag{B.10}$$

where $\sigma_{ij} \equiv \frac{1}{2i}[\sigma_i, \sigma_j] = \epsilon_{ijk}\sigma_k$. This is called the Yang monopole [52]. This is singular at the south pole $x_5 = -r$. We have introduced the t'Hooft symbol;

$$\eta^i_{\alpha\beta} = \epsilon_{i\alpha\beta 4} - \delta_{i\alpha}\delta_{4\beta} + \delta_{i\beta}\delta_{4\alpha}\,, \tag{B.11}$$

where the nonzero values are $\eta^1_{23} = \eta^1_{41} = \eta^2_{31} = \eta^2_{42} = \eta^3_{12} = \eta^3_{43} = 1$. We note that $\Sigma^N_{\mu\nu} \equiv i\eta^i_{\mu\nu}\sigma_i$ satisfy

$$[\Sigma^N_{\mu\nu}, \Sigma^N_{\lambda\rho}] = 2\left(\delta_{\nu\lambda}\Sigma^N_{\mu\rho} + \delta_{\mu\rho}\Sigma^N_{\nu\lambda} - \delta_{\mu\lambda}\Sigma^N_{\nu\rho} - \delta_{\nu\rho}\Sigma^N_{\mu\lambda}\right). \tag{B.12}$$

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$\Sigma^N_{\mu\nu}$ is actually a left upper part of the matrix $\Gamma_{\mu\nu}$ ($\mu, \nu = 1, \ldots, 4$), where Γ_μ is given by (B.8). We define the field strength as follows,

$$\begin{aligned} F_{\mu\nu} &\equiv -i[D_\mu, D_\nu] \equiv -i[\partial_\mu + iA_\mu, \partial_\nu + iA_\nu] \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]\,. \end{aligned} \tag{B.13}$$

The field strength for the Yang monopole (B.10) is

$$\begin{aligned} F^N_{5\alpha} &= -\frac{r+x_5}{r^2} A_\alpha \\ F^N_{\alpha\beta} &= \frac{1}{r^2}\left(A_\alpha x_\beta - A_\beta x_\alpha + \frac{1}{2}\eta^i_{\alpha\beta}\sigma_i\right), \end{aligned} \tag{B.14}$$

where $\alpha, \beta = 1, \ldots, 4$.

We next consider the monopoles on S^6 and S^8. As a generalization of the monopoles on S^2 and S^4, it is natural to start with the following projection operator

$$P^{(k)} \equiv \frac{1}{2}(1_{2^k} + n_\mu \Gamma_\mu) = \begin{pmatrix} 1 + n_{2k+1} & n_{2k} - i n_i \gamma_i \\ n_{2k} + i n_i \gamma_i & 1 - n_{2k+1} \end{pmatrix}, \tag{B.15}$$

where $k = 3, 4$. An explicit form of the $2^k \times 2^k$ $(2k+1)$-dimensional gamma matrix is

$$\begin{aligned} \Gamma_i &= \begin{pmatrix} 0 & -i\gamma_i \\ i\gamma_i & 0 \end{pmatrix}, \qquad (i = 1, \ldots, 2k-1) \\ \Gamma_{2k} &= \begin{pmatrix} 0 & 1_{2^{k-1}} \\ 1_{2^{k-1}} & 0 \end{pmatrix}, \qquad \Gamma_{2k+1} = \begin{pmatrix} 1_{2^{k-1}} & 0 \\ 0 & -1_{2^{k-1}} \end{pmatrix}, \end{aligned} \tag{B.16}$$

where γ_i is the $(2k-1)$-dimensional gamma matrix. This map was used in [32] to construct a higher dimensional quantum Hall system. One may notice that the map for $k = 4$ case is different from the third Hopf map. The calculation of the Berry phase introduces the following $SO(2k)$ monopole field on S^{2k}:

$$\begin{aligned} A_N = A^\mu_N dx_\mu &= -i\langle v|d|v\rangle = -i\begin{pmatrix} 1 & 0 \end{pmatrix} \frac{P^{(k)}}{N^{(k)}} d\left(\frac{P^{(k)}}{N^{(k)}}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{2r(r+x_{2k+1})}\left(-i\gamma_{ij}x_i dx_j + \gamma_i x_{2k} dx_i - \gamma_i x_i dx_{2k}\right) \\ &= \frac{-i}{2r(r+x_{2k+1})} \Sigma^N_{\alpha\beta} x_\alpha dx_\beta \end{aligned} \tag{B.17}$$

where $\Sigma^N_{\alpha\beta} = (\Sigma^N_{ij}, \Sigma^N_{2k,i}) \equiv (\gamma_{ij}, i\gamma_i) = (\frac{1}{2}[\gamma_i, \gamma_j], i\gamma_i)$ is the spinor representation of $SO(6)$, satisfying the algebra of (B.12). $\Sigma^N_{\alpha\beta}$ is a subspace of $\Gamma_{\alpha\beta}$ ($\alpha, \beta = 1, \ldots, 2k$) which is labelled by $\Gamma_{2k+1} = 1$, where Γ_μ is the $(2k+1)$-dimensional gamma matrix (B.16). The gauge field strength corresponding to the above monopole gauge field is

$$\begin{aligned} F^N_{2k+1\alpha} &= -\frac{r+x_{2k+1}}{r^2} A_\alpha \\ F^N_{\alpha\beta} &= \frac{1}{r^2}\left(A_\alpha x_\beta - A_\beta x_\alpha - \frac{i}{2}\Sigma^N_{\alpha\beta}\right), \end{aligned} \tag{B.18}$$

where $\alpha, \beta = 1, \ldots, 2k$. These monopole fields have the $SO(2k+1)$ symmetry.

At the north pole ($x_\alpha = 0,\ x_{2k+1} = r$), the field strength becomes $F_{2k+1\alpha} = 0$, $F_{\alpha\beta} = -\frac{i}{2r^2}\Sigma^N_{\alpha\beta}$. The field strength on S^{2k} ($k = 2, 3, 4$) at the north pole satisfies the following relations,

$$\begin{aligned} \epsilon_{\alpha\beta\gamma\delta} F^{\gamma\delta} &= -2F_{\alpha\beta}\,, \\ \epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6} F^{\alpha_3\alpha_4} F^{\alpha_5\alpha_6} &= \frac{4}{r^2}(n+2) F_{\alpha_1\alpha_2}\,, \\ \epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7\alpha_8} F^{\alpha_3\alpha_4} F^{\alpha_5\alpha_6} F^{\alpha_7\alpha_8} &= -\frac{12}{r^4}(n+2)(n+4) F_{\alpha_1\alpha_2}\,. \end{aligned} \tag{B.19}$$

The first one is a (anti)self-dual relation of the instanton, and the second and the third ones are considered as a higher dimensional generalization of the self-dual relation.

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